Discordant Uranium-Lead Ages, I

GEORGE W. WETHERILL

Abstract—A graphical procedure is described for rapid calculation of discordant uranium-lead ages resulting from multiple episodes of uranium-lead fractionation. A proof of the validity of this graphical procedure is given. The graphical procedure is extended to permit the calculation of the effect of the presence of primary radiogenic lead and of constant loss of intermediate daughter products.

Introduction—The element uranium has two long-lived isotopes, U^{238} and U^{235}, the final decay products of which are Pb^{206} and Pb^{207}, respectively. Measurement of the uranium concentration and the concentration of radiogenic Pb^{206} and Pb^{207} in a chemical system (such as a mineral) containing uranium permits the calculation of two uranium-lead ages, the U^{238}-Pb^{206} age, T_1 and the U^{235}-Pb^{207} age, T_2 calculated from the equations of radioactive decay

$$\left.\begin{aligned} T_1 &= \frac{1}{\lambda_{U^{238}}} \ln\left(\frac{Pb^{206}}{U^{238}} + 1\right) \\ T_2 &= \frac{1}{\lambda_{U^{235}}} \ln\left(\frac{Pb^{207}}{U^{235}} + 1\right) \end{aligned}\right\} \quad (1)$$

In this discussion it will be convenient to designate the U^{238}-Pb^{206} decay by superscript or subscript 1, and the U^{235}-Pb^{207} decay by the subscript 2. Thus, equations (1) become

$$\left.\begin{aligned} T_1 &= \frac{1}{\lambda_1} \ln\left(\frac{D_1}{P_1} + 1\right) \\ T_2 &= \frac{1}{\lambda_2} \ln\left(\frac{D_2}{P_2} + 1\right) \end{aligned}\right\} \quad (1a)$$

where the λ's are the decay constants, and D and P refer to the daughter and parent concentrations, respectively.

Where these two ages are found to be equal to one another, the ages are said to be 'concordant.' When they are unequal, they are said to be 'discordant.'

These two calculated ages will be equal to one another and to the true age of the mineral if the following assumptions are fulfilled: (a) There have been no gains or losses of uranium or lead during the time since the formation of the system. (b) There have been no gains or losses of intermediate members of the radioactive decay scheme, for example, radon, or ionium. (c) Proper corrections have been made for the initial concentration of Pb^{206} and Pb^{207}. (d) The chemical analyses have been properly performed and the correct decay constants λ_1 and λ_2 have been used. When these assumptions have been fulfilled, the ages will be concordant; when they are not fulfilled, the ages will be either discordant or 'accidentally' concordant.

These papers will discuss discordance and accidental concordance arising from failure of assumptions (a), (b), and (c), which may be considered 'intrinsic discordance' as opposed to 'technical' discordance resulting from failure of assumption (d).

In this first paper a graphical scheme will be presented for the calculation of the effects of failure of assumption (a) at discreet episodes in the history of the mineral, and a discussion will be given of the effects of failure of assumptions (b) and (c), within the framework of this graphical method. In a subsequent publication this graphical procedure will be applied to problems of geochronology, namely, the interpretation of regularities in a group of discordant ages such as those by *Ahrens* [1955] as well as the inference of the true age of a group of cogenetic minerals giving discordant ages, even when no regularities are present. This graphical scheme will make use of a diagram (Fig. 1) in which the mole ratio Pb^{206}/U^{238} (D_1/P_1) is plotted as the ordinate and the mole ratio Pb^{207}/U^{235} (D_2/P_2) as the abscissa. In the case of concordant ages, for every age $\tau_0 = T_1 = T_2$ there will correspond unique values of D_1/P_1 and D_2/P_2 defined by the equations

$$\left.\begin{aligned} \frac{D_1}{P_1} &= e^{\lambda_1\tau_0} - 1 \\ \frac{D_2}{P_2} &= e^{\lambda_2\tau_0} - 1 \end{aligned}\right\} \quad (2)$$

The locus of these values for $0 < \tau_0 < \infty$ is the curve marked 'concordia' on Figure 1.

Description of the graphical procedure—The procedure will be described without proof. The proof of its validity is found in the following section. Consider a sample of uranium mineral (or more generally a chemical system containing uranium) having a true age τ_0. If the assumptions

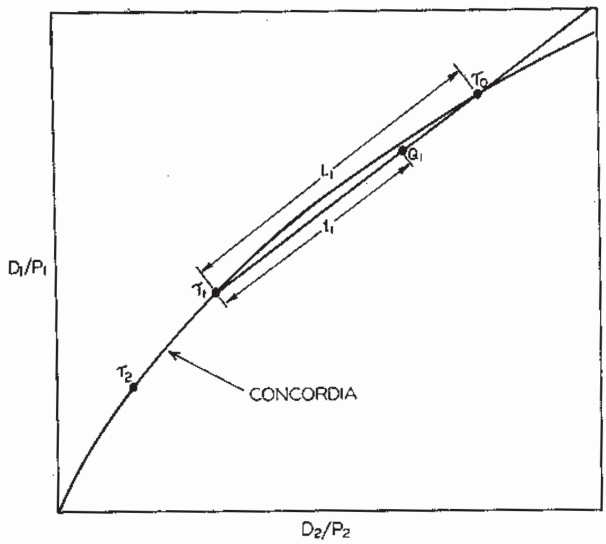

FIG. 1 – D/P diagram illustrating the graphical procedure for calculating the uranium-lead ages resulting from a given history of uranium-lead fractionation.

(a), (b), (c), and (d) are valid, the point characteristic of this mineral sample will lie on concordia at τ_0. If the mineral has lost lead or uranium or gained uranium during a geologically brief episode at a time τ_1 years ago, the position of the point (Q_1) characteristic of this mineral may be found by the following procedure: (1) Draw a straight line passing through the point on the curve 'concordia' corresponding to a true age τ_0 and that corresponding to a true age τ_1. (2) Defining the length of the straight segment $\overline{\tau_0\tau_1} = L_1$, measure off a distance along this segment of length $l_1 = R_1L_1$ from τ_1. R_1 is the ratio by which D_1/P_1 and D_2/P_2 changed at time τ_1. That is

$$R_1 = (D/P)_{\text{immediately after loss}}/(D/P)_{\text{immediately before loss}}$$

The end of this segment is the desired point Q_1. In the case of lead loss, it is assumed that R is the same for both the lead isotopes, that is, that the lead which is lost has the same isotopic composition as the total lead which was present in the total lead which was present in the mineral. This is not a very severe restriction for lead loss, but it must be noted that lead addition will not in general fulfill this condition, and therefore the graphical procedure has been limited to losses of lead or uranium or additions of uranium. This case where R is unequal for D_1/P_1 and D_2/P_2 is discussed in the section on *Extensions of the graphical procedure*. For example, if half the lead were lost with no loss of uranium, the $R_1 = \frac{1}{2}$. If, on the other hand, $\frac{1}{3}$ of the uranium were lost, $R_1 = \frac{3}{2}$, and $l_1 = \frac{3}{2}L_1$.

From the procedure given above, it is seen that regardless of how much uranium or lead is lost during this episode, the resulting point (Q_1) will lie on this straight line. From the coordinates of the point (D_1/P_1, D_2/P_2) found by this procedure, the discordant uranium-lead ages can be calculated by use of Eq. (2). Thus the effect of a single episode of uranium-lead fractionation has been determined.

For the special case $R_1 = 1$, that is, no fractionation, the result is that the point Q_1 characterizing the mineral remains on the curve 'concordia' at τ_0. If, on the other hand, all the lead were lost at τ_1, perhaps due to a remineralization with exclusion of lead, then $R_1 = 0$ and $l_1 = 0$ and the point characterizing the sample will lie on the curve 'concordia' at τ_1. In this case the age of the sample can be considered more properly to be τ_1.

For the case of multiple episodes of uranium-lead fractionation, the resulting point Q_n can be found by extension of this same procedure. Assume a second fractionation R_2 occurred at a later time τ_2. The effect of this is found by the following procedure: (1) Draw a straight line between the point Q_1 (found by the above procedure for the first fractionation) and the point τ_2 on concordia. (2) Defining the length of the straight segment $\overline{Q\tau_2} = L_2$, the desired point Q_2 will lie at a distance $l_2 = R_2L_2$ from τ_2 on concordia. For a third fractionation at τ_3, the procedure is repeated with $\overline{Q_2\tau_3} = L_3$ and $l_3 = R_3L_3$ giving point Q_3. For the case of n fractionations, the procedure is repeated n times, finally resulting in a point Q_n. From the coordinates of this point, the discordant ages T_1 and T_2 can be calculated. These are the ages that would be measured for a mineral with a true age τ_0 that has undergone n fractionations R_i, ($i = 1, 2, \cdots n$) at times τ_1 in the past.

This construction is illustrated by the following example, illustrated by Figure 2. Consider a uranium mineral having a true age of 1350 million years. 900 million years ago the uranium and lead within this mineral were partially separated. At this time 17 pct of the lead within the mineral was lost, while at the same time 50 pct of the uranium was lost. As a consequence of this fractionation, the ratios D_1/P_1 and D_2/P_2 change by a factor 1.65 and $R_1 = 1.65$. In recent times (essentially zero million years ago) a second fractionation occurred, resulting in the loss of 27 pct of the lead present in the mineral, with a loss of

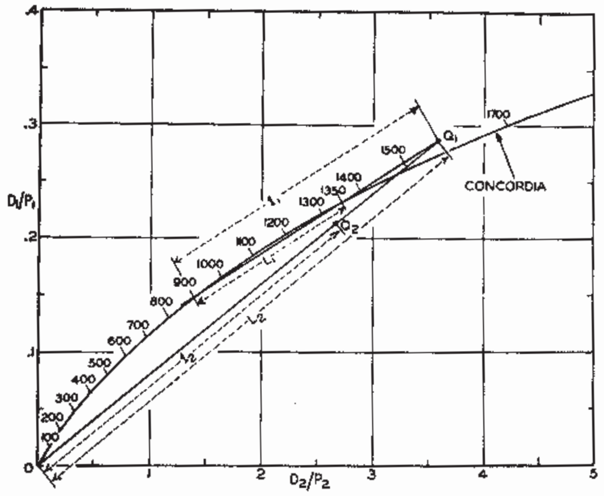

FIG. 2 – D/P diagram illustrating a numerical example of the graphical procedure

only one per cent of the uranium. Hence R_2 = 0.74.

Following the graphical procedure described above, a straight line is drawn through the points 900 and 1350 on the curve 'concordia.' The point Q_1 is found by measuring the separation of these two points and measuring off a distance 1.65 times this separation along this line from the point 900. Q_2 is found by drawing a similar line between 0 and Q_1, and measuring off the proper distance from 0. This distance is 0.74 times the distance between 0 and Q_1.

Since this hypothetical mineral underwent two fractionations, it is characterized by the point Q_2. From the coordinates of this point (D_1/P_1 = 0.214, and D_2/P_2 = 2.64) the ages T_1 = 1260 *m y* and T_2 = 1330 *m y* are calculated from (1a). An exact analytic calculation of this example gives T_1 = 1250 and T_2 = 1330 *m y*. The accuracy of this graphical procedure depends only on the care with which the construction is made.

The effect of a continuous process can be approximated by graphically calculating the continuous process as a sum of episodic fractionations with the interval between episodes small.

Proof of the graphical procedure—The graphical procedure outlined above will be proved to be correct by showing that the coordinates of the resulting point Qn corresponds to the coordinates D_1/P_1 and D_2/P_2 which would be found by an analytic calculation of the effect of n fractionations $R_i(i = 1, 2, \cdots n)$ at times τ_1 in the past. In the following discussion the symbol t will be used to indicate time increasing in the usual sense and having the value zero at the time of mineralization. The symbol τ_1 will be used to indicate the time of an event measured back from the present.

This analytic expression will be obtained by solution of the equations of radioactive decay generalized to include the effects of gains or losses of parent and daughter isotopes. These are

$$\left.\begin{aligned} \frac{dP}{dt} &= -\lambda P + G_P P \\ \frac{dD}{dt} &= \lambda P + G_D D \end{aligned}\right\} \quad (3)$$

where P and D refer to the concentrations of parent and radiogenic daughter isotopes, λ is the decay constant = ln 2/half-life, G_P and G_D are arbitrary functions of times representing gains, ($G > 0$) or losses ($G < 0$) of parent and daughter isotopes, respectively. These equations may be solved by elementary methods with the result

$$\frac{D}{P} = \lambda e^{F(\tau_0)} \int_0^{\tau_0} e^{-F}\, dt \quad (4)$$

where

$$F = \lambda t + \int (G_D - G_P)\, dt,$$

and τ_0 is the true age of the mineral. (An equation essentially the same as (4) has also been derived by F. Wickman and his discussion of it will be published in the report of the Pennsylvania State University Conference on Nuclear Geophysics, September, 1955).

It may be seen by inspection of the form of F in (4) that a gain of parent represented by a function G_P will have exactly the same effect on the ratio D/P as a loss of daughter with $G_D = -G_P$. It will therefore be impossible to distinguish between loss of parent and gain of daughter or between gain of parent and loss of daughter by the effect on the ratio D/P. Without any loss of generality the expression ($G_D - G_P$) can be replaced by an arbitrary function of time G, which will represent the net effect of gains or losses of parent and daughter. In this work where 'loss of daughter' is used, it will be understood that the possibility of gain of parent is also implied, and similarly for loss of parent.

By assuming different forms of G, the effects of different kinds of fractionation processes can be calculated. For the case of n episodic fractionations at times τ_i, $i = 1, \cdots n$ the function G will be given by

$$G = \sum_{i=1}^{n} a_i \delta[t - (\tau_0 - \tau_i)] \quad (5)$$

where $\delta[t - (\tau_0 - \tau_i)]$ is the Dirac δ-function [*Dirac*, 1949] defined by

$$\left.\begin{aligned} \delta(x) &= 0 \quad x \neq 0 \\ \int_{-a}^{a} \delta(x) &= 1 \quad a > 0 \end{aligned}\right\} \qquad (6)$$

Thus the δ-function can be visualized as a function which is zero everywhere except in the immediate neighborhood of one point. Similarly, G will be zero except at the moments τ_i. The difficulties of mathematical rigor associated with the use of (6) can be resolved in the usual way by replacing the δ-function by a gaussian and taking the standard deviation to be arbitrarily small. Thus the fractionation can be thought of as taking place over a period of time which is geologically short, for example, a million years.

By substitution of G (5) into (4) and integration, one obtains

$$\frac{D}{P} = \sum_{j=1}^{n} e^{\sum_{i=j}^{n} a_i} [e^{\lambda \tau_{j-1}} - e^{\lambda \tau_j}] + e^{\lambda \tau_n} - 1 \qquad (7)$$

The differential equations (3) can be combined in the form

$$\frac{d}{dt}\left(\frac{D}{P}\right) = \lambda + \left(\frac{D}{P}\right)(G + \lambda) \qquad (8)$$

or

$$\frac{d}{dt} \ln \left(\frac{D}{P}\right) = \lambda \frac{P}{D} + G + \lambda \qquad (9)$$

This equation will be integrated over a short period of time (2ϵ) including one of the episodes τ_i.

$$\int_{\tau_0-(\tau_i+\epsilon)}^{\tau_0-(\tau_i-\epsilon)} \frac{d}{dt} \ln \left(\frac{D}{P}\right) dt = \lambda \int_{\tau_0-(\tau_i+\epsilon)}^{\tau_0-(\tau_i-\epsilon)} \frac{P}{D}\, dt + \int_{\tau_0-(\tau_i+\epsilon)}^{\tau_0-(\tau_i-\epsilon)} \Sigma a_i \delta(t - (\tau_0 - \tau_i))\, dt + \lambda t \Big|_{\tau_0-(\tau_i+\epsilon)}^{\tau_0-(\tau_i-\epsilon)} \qquad (10)$$

or

$$\ln \frac{\left(\frac{D}{P}\right)_{\tau_i-\epsilon}}{\left(\frac{D}{P}\right)_{\tau_i+\epsilon}} = \lambda \int_{\tau_0-(\tau_i+\epsilon)}^{\tau_0-(\tau_i-\epsilon)} \frac{P}{D}\, dt + a_i + 2\epsilon\lambda \qquad (11)$$

as $\epsilon \to 0$

$$\ln \frac{\left(\frac{D}{P}\right)_{\text{after loss}}}{\left(\frac{D}{P}\right)_{\text{before loss}}} \to a_i \qquad (12)$$

then

$$\frac{\left(\frac{D}{P}\right)_{\text{immediately after loss}}}{\left(\frac{D}{P}\right)_{\text{immediately before loss}}} = e^{a_i} \qquad (13)$$

or in accordance with the definition of R_i

$$R_i = e^{a_i} \qquad (14)$$

(The proof of (14) has been materially simplified as a result of a suggestion by G. Wasserburg.)

Eq. (7) can now be rewritten as

$$\frac{D}{P} = \sum_{j=1}^{n} [e^{\lambda \tau_{j-1}} - e^{\lambda \tau_j}] \prod_{i=j}^{n} R_i + (e^{\lambda \tau_n} - 1) \qquad (15)$$

This is the general expression for the ratio D/P resulting from the decay of parent in a mineral of true age τ_0 which has undergone n fractionations, R_i at times τ_i. For a uranium mineral there will be two such expressions, one giving D_1/P_1 and the other D_2/P_2.

Using this general expression, the graphical procedure will be proved by induction. (a) It will first be shown that the point $(D_1/P_1, D_2/P_2)$ found by the graphical procedure agrees with the result of the analytic calculation for the case $n = 1$. (b) It then will be shown that if the construction is valid for $n = m$, it will also be valid for $n = m + 1$.

Demonstration of (a) and (b) above will prove the construction valid for any number of fractionations.

Regarding (a), by substituting $n = 1$ into (15) we obtain

$$\left.\begin{aligned} \frac{D_1}{P_1} &= R_1[e^{\lambda_1 \tau_0} - e^{\lambda_1 \tau_1}] + e^{\lambda_1 \tau_1} - 1 \\ \frac{D_2}{P_2} &= R_1[e^{\lambda_2 \tau_0} - e^{\lambda_2 \tau_1}] + e^{\lambda_2 \tau_1} - 1 \end{aligned}\right\} \qquad (16)$$

These are the parametric equations for the locus of points representing minerals having a true age τ_0 and which underwent a fractionation R_1 at a time τ_1 in the past. According to the graphical construction, this locus should be a straight line passing through τ_0 and τ_1 on the curve 'concordia.'

That it passes through τ_0 can be seen by letting the parameter $R_1 = 1$, then

$$\left.\begin{aligned} \frac{D_1}{P_1} &= e^{\lambda_1 \tau_0} - 1 \\ \frac{D_2}{P_2} &= e^{\lambda_2 \tau_0} - 1 \end{aligned}\right\} \qquad (17)$$

which are the coordinates of τ_0 on the curve 'concordia' as given by (2).

Similarly when the parameter $R_1 = 0$

$$\left.\begin{aligned} \frac{D_1}{P_1} &= e^{\lambda_1\tau_1} - 1 \\ \frac{D_2}{P_2} &= e^{\lambda_2\tau_1} - 1 \end{aligned}\right\} \quad (18)$$

which are the coordinates of τ_1 on concordia. By differentiating each of (16) with respect to the parameter R_1 and dividing, we obtain the slope of the locus

$$\frac{d\left(\frac{D_1}{P_1}\right)}{d\left(\frac{D_2}{P_2}\right)} = \frac{e^{\lambda_1\tau_0} - e^{\lambda_1\tau_1}}{e^{\lambda_2\tau_0} - e^{\lambda_2\tau_1}} \quad (19)$$

which is independent of R_1, that is of the position of the point on the locus. The slope is therefore the same at all points on the locus, that is the line is straight.

According to the graphical construction, the distance along this line (l) between τ_1 on concordia and the point characterizing a mineral which underwent a fractionation R_1 at τ_1 is given by $l_1 = R_1L_1$ where L_1 is the length of the segment $\overline{\tau_0\tau_1}$. The length can be found by use of the coordinates of τ_0 and τ_1 on concordia from equations (17) and (18) and the Pythogorean theorem to be

$$L_1^2 = [e^{\lambda_1\tau_0} - e^{\lambda_1\tau_1}]^2 + [e^{\lambda_2\tau_0} - e^{\lambda_2\tau_1}]^2 \quad (20)$$

By using (16) and (18) and the Pythogorean theorem we obtain similarly

$$l_1^2 = R_1^2[e^{\lambda_1\tau_0} - e^{\lambda_1\tau_1}]^2 + R_1^2[e^{\lambda_2\tau_0} - e^{\lambda_2\tau_1}]^2 \quad (21)$$

and by comparing (20) and (21) we obtain

$l_1 = R_1L_1$ thus completing the proof for $n = 1$. (22)

Regarding (b) if it is assumed that the graphical construction is valid for $n = m$, it will now be shown that it is valid for $n = m + 1$.

The coordinates of a point which has undergone $n = m + 1$ fractionations will be

$$\left.\begin{aligned} \frac{D_1}{P_1} &= \sum_{j=1}^{m+1} [e^{\lambda_1\tau_{j-1}} - e^{\lambda_1\tau_j}] \prod_{i=j}^{m+1} R_i + (e^{\lambda_1\tau_{m+1}} - 1) \\ \frac{D_2}{P_2} &= \sum_{j=1}^{m+1} [e^{\lambda_2\tau_{j-1}} - e^{\lambda_2\tau_j}] \prod_{i=j}^{m+1} R_i + (e^{\lambda_2\tau_{m+1}} - 1) \end{aligned}\right\} \quad (23)$$

For the case $R_{m+1} = 1$, that is, no fractionation at stage $(m + 1)$ the point will have the coordinates

$$\begin{aligned} \frac{D_1}{P_1} &= \sum_{j=1}^{m} [e^{\lambda_1\tau_{j-1}} - e^{\lambda_1\tau_j}] \prod_{i=1}^{m} R_i \\ &\quad + (e^{\lambda_1\tau_m} - e^{\lambda_1\tau_{m+1}}) + (e^{\lambda_1\tau_{m+1}} - 1) \quad (24) \\ &= \sum_{j=1}^{m} [e^{\lambda_1\tau_{j-1}} - e^{\lambda_1\tau_j}] \prod_{i=j}^{m} R_i + (e^{\lambda_1\tau_m} - 1) \end{aligned}$$

and similarly

$$\frac{D_2}{P_2} = \sum_{j=1}^{m} [e^{\lambda_2\tau_{j-1}} - e^{\lambda_2\tau_j}] \prod_{i=j}^{m} R_i + (e^{\lambda_2\tau_m} - 1)$$

which according to (15) are the coordinates of a point Q_m which has undergone m fractionations R_i at times τ_i. Thus the locus of a point which has undergone $m + 1$ fractions R_i at times τ_i passes through this point. For the case $R_{m+1} = 0$, we get the point

$$\frac{D_1}{P_1} = e^{\lambda_1\tau_{m+1}} - 1 \quad (25)$$

and

$\frac{D_2}{P_2} = e^{\lambda_2\tau_{m+1}} - 1$, that is, the point τ_{m+1} on concordia

By differentiating (23) with respect to R_{m+1} and taking their ratio we obtain

$$\frac{d\left(\frac{D_1}{P_1}\right)}{d\left(\frac{D_2}{P_2}\right)} = \frac{\sum_{j=1}^{m} [e^{\lambda_1\tau_{j-1}} - e^{\lambda_1\tau_j}] \prod_{i=j}^{m} R_i + (e^{\lambda_1\tau_m} - e^{\lambda_1\tau_{m+1}})}{\sum_{j=1}^{m} [e^{\lambda_2\tau_{j-1}} - e^{\lambda_2\tau_j}] \prod_{i=j}^{m} R_i + (e^{\lambda_2\tau_m} - e^{\lambda_2\tau_{m+1}})} \quad (26)$$

which is independent of R_{m+1}. Thus the locus of points corresponding to minerals which have undergone varying fractionations R_{m+1} at time τ_{m+1} is a straight line passing through Q_m and τ_{m+1} on concordia, as given by the graphical procedure.

The distance L_{m+1} between Q_m and τ_{m+1} on concordia will be (by use of the Pythogorean theorem)

$$\begin{aligned} L^2_{m+1} = &\left\{\sum_{j=1}^{m} [e^{\lambda_1\tau_{j-1}} - e^{\lambda_1\tau_j}] \prod_{i=j}^{m} R_i + (e^{\lambda_1\tau_m} - 1) - (e^{\lambda_1\tau_{m+1}} - 1)\right\}^2 \\ &+ \left\{\sum_{j=1}^{m} [e^{\lambda_2\tau_{j-1}} - e^{\lambda_2\tau_j}] \prod_{i=j}^{m} R_i + (e^{\lambda_2\tau_m} - 1) - (e^{\lambda_2\tau_{m+1}} - 1)\right\}^2 \quad (27) \end{aligned}$$

collecting terms

$$L^2_{m+1} = \left\{ \sum_{j=1}^{m+1} [e^{\lambda_1 \tau_{j-1}} - e^{\lambda_1 \tau_j}] \frac{\prod_{i=j}^{m+1} R_i}{R_{m+1}} \right\}^2 + \left\{ \sum_{j=1}^{m+1} [e^{\lambda_2 \tau_{j-1}} - e^{\lambda_2 \tau_j}] \frac{\prod_{i=j}^{m+1} R_i}{R_{m+1}} \right\}^2 \quad (28)$$

The distance l_{m+1} between τ_{m+1} on the curve 'concordia' and the point characteristic of a mineral which has undergone $m + 1$ fractionations R_i at times τ_i will be

$$l^2_{m+1} = \left\{ \sum_{j=1}^{m+1} [e^{\lambda_1 \tau_{j-1}} - e^{\lambda_1 \tau_j}] \prod_{i=j}^{m+1} R_i \right\}^2 + \left\{ \sum_{j=1}^{m+1} [e^{\lambda_2 \tau_{j-1}} - e^{\lambda_2 \tau_j}] \prod_{i=j}^{m+1} R_i \right\}^2 \quad (29)$$

By comparison of (28) and (29)

$$l^2_{m+1} = R^2_{m+1} L^2_{m+1}$$

Thus the locus of points characteristic of minerals which have undergone m fractionations R_i at times τ_i and an additional fractionation R_{m+1} at time τ_{m+1} will be a straight line passing through the point Q_m and the point τ_{m+1} on concordia. For a given fractionation R_{m+1}, the distance of the point from τ_{m+1} on concordia will be equal to $R_{m+1}L_{m+1}$ where L_{m+1} is the distance from Q_m to τ_{m+1} on concordia. By hypothesis the point Q_m (which represents the point characteristic of minerals which have undergone m fractionations R_i, $i = 1, 2, \cdots m$ at times τ_i) is given correctly by the graphical procedure. It has been shown above that the analytical calculation is in agreement with the graphical procedure for finding Q_{m+1} from Q_m. Therefore, the point Q_{m+1} found by the analytical calculation will be the same point found by the graphical procedure, thus completing the proof.

Extensions of the graphical procedure—In the foregoing discussion a procedure has been demonstrated for graphically calculating the discordant uranium-lead ages which will result when a mineral undergoes a series of episodes of uranium-lead fractionation (failure of assumption a). A brief discussion will now be given of the calculation of the effects of failure of assumptions (b), and (c) as well as the effect of unequal fractionation of Pb^{206} and Pb^{207}.

Loss of intermediate decay products—For the equilibrium case (implicit in (1)) the loss of an intermediate decay product is equivalent to the decay constant λ for the growth of daughter being less than the decay constant λ for the decay of parent (3).

Therefore, (3) can be replaced by

$$\left.\begin{aligned} \frac{dP}{dt} &= -\lambda P + G_P P \\ \frac{dD}{dt} &= \lambda \Lambda(t) P + G_D D \end{aligned}\right\} \quad (30)$$

where $\Lambda(t)$ represents the fraction of intermediate decay product which is retained.

These equations can be integrated to give

$$\frac{D}{P} = \lambda e^{F(\tau_0)} \int_0^{\tau_0} \Lambda(t) e^{-F} \, dt \quad (31)$$

It may be expected that Λ will be different for the two decay systems, U^{238} and U^{235}.

For the case of constant loss of intermediate decay product the function Λ can be taken outside the integral. In this case the value of D/P will be decreased by a factor Λ. For combined multiple fractionation and constant intermediate product loss, the resulting values of D_1/P_1 and D_2/P_2 can be found by first applying the graphical procedure and finally multiplying D_1/P_1 by Λ_1, and D_2/P_2 by Λ_2.

Presence of primary radiogenic lead—Eq. (3) can be generalized to include the effects of primary radiogenic lead with the result

$$\frac{D}{P} = e^{F(\tau_0)} \left[\left(\frac{D}{P}\right)_0 e^{-F(0)} + \int_0^{\tau_0} \lambda e^{-F} \, dt \right] \quad (32)$$

where $(D/P)_0$ is the initial radiogenic daughter to parent ratio.

For the case of multiple episodic fractionations

$$\left(\frac{D}{P}\right) \sum_{j=1}^{n} [e^{\lambda \tau_{j-1}} - e^{\lambda \tau_j}] \prod_{i=j}^{n} R_i + (e^{\lambda \tau_n} - 1) + \left(\frac{D}{P}\right)_0 e^{\lambda \tau_0} \prod_{i=1}^{n} R_i \quad (33)$$

which differs from (15) only in the last term. Therefore, the effect of primary radiogenic daughter can be found by applying the graphical procedure and finally adding the term

$$\left(\frac{D}{P}\right)_0 e^{\lambda \tau_0} \prod_{i=1}^{n} R_i \quad \text{to} \quad \frac{D_1}{P_1} \quad \text{and} \quad \frac{D_2}{P_2} \quad (34)$$

By a proof along the lines of that in *Proof of the graphical procedure*, it can also be shown that the graphical calculation is valid for the case of primary radiogenic daughter if the starting point for the graphical procedure is taken to the coordinates of τ_0 on concordia augmented by

$$\left(\frac{D}{P}\right)_0 e^{\lambda\tau_0}$$

The effect of unequal fractionation of Pb^{206} *and* Pb^{207}—As a consequence of an earlier process of uranium-lead fractionation, the uranium in a mineral may be displaced from the lead. The radiogenic lead resulting from the decay of this uranium will then be in the vicinity of the uranium and the two lead isotopes will not be homogeneously distributed within the mineral. Any subsequent fractionation might then cause D_1/P_1 and D_2/P_2 to change by different factors.

In this case it can be shown that the points Q_{m+1} resulting from varying amounts of fractionation of this type will not lie on a straight line between Q_m and τ_{m+1} on concordia but will lie on a curved line between these two points. The coordinate D_1/P_1 for Q_{m+1} will be

$$(e^{\lambda_1\tau_{m+1}} - 1) + R'_{m+1}\left[\left(\frac{D_1}{P_1}\right)_{Q_m} - (e^{\lambda_1\tau_{m+1}} - 1)\right] \quad (35)$$

where R'_{m+1} is the factor by which D_1/P_1 is changed. A similar expression will give (D_2/P_2) for Q_{m+1}.

Concluding remarks—A graphical procedure has been demonstrated for calculating the effects of failure of assumptions (a), (b), and (c). By use of this procedure the discordant ages resulting from a given history can be calculated uniquely. However, if the discordant ages are given, it is not possible to infer the history uniquely. This graphical procedure will be found useful, however, in inferring the possible histories of a given geological unit and in some cases the probable history. These applications and their limitations will be discussed in a subsequent publication.

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Department of Terrestrial Magnetism, Carnegie Institution of Washington, Washington 15, D. C.

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